Classical Electromagnetism

Maxwell's equations are

$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0 \qquad \nabla \cdot \mathbf{B} = 0 \qquad \nabla \times \mathbf{B} - (\partial \mathbf{E}/\partial t)/c^2 = \mu_0 \mathbf{J} \qquad \nabla \times \mathbf{E} + \partial \mathbf{B}/\partial t = 0$$

where

J = current density (amps passing thru a surface)
ρ = charge density (electrons per cubic meter)
E = electric field
B = magnetic field

Which (in order) *simplistically* say that

- The strength of the electric field is proportional to the charge density
- There are no separate magnetic "charges" (monopoles) – north and south must come in pairs
- The change of a magnetic field in space is proportional to the change of an electric field in time
- The change of an electric field in space is proportional to the change of a magnetic field in time

There is also a **continuity equation** : $(\partial\rho/\partial t) = -\nabla \cdot \mathbf{J}$ which says that the amount of current is proportional to the change in charge over time (this is also sometimes called "**conservation of charge**").

The Lorentz force on a charged particle due to electric and magnetic fields is : $\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B})$

It is often easier to use two other quantities :

φ = "**scalar potential**", "**electric scalar potential**", "**electric potential**"
A = "**vector potential**", "**magnetic vector potential**", "**magnetic potential**"

Which are defined by :

$$\mathbf{A}(p_1, t) = \frac{\mu_0}{4\pi} \int_{V_2} \frac{\mathbf{j}(p_2, t_r)}{r_{12}} \mathrm{d}V \qquad \phi(p_1, t) = \frac{1}{4\pi\epsilon_0} \int_{V_2} \frac{\rho(p_2, t_r)}{r_{12}} \mathrm{d}V$$

where

t is the time at which the value of **A** and φ are to be calculated
$\mathbf{p_1}$ is the point at which the value of **A** and φ are to be calculated
$\mathbf{p_2}$ is a point at which the value of **J** or ρ or both are non-zero at least some of the time
r_{12} is the distance from point $\mathbf{p_1}$ to point $\mathbf{p_2}$
$t_r = t - r_{12}/c$ is a time earlier than t (by r_{12}/c) which is the time it takes an effect generated at $\mathbf{p_2}$ to propagate to $\mathbf{p_1}$ at the speed of light; t_r is also called *retarded time*
V_2 is the volume of all points $\mathbf{p_2}$ where **J** or ρ is non-zero at least sometimes

Note that the components of **A** depend only on the components of **J** in the same direction. If a current is carried in a long straight wire, **A** points in the same direction as the wire.

Then E and B are given by :

$$\mathbf{E} = -\nabla\varphi - \partial\mathbf{A}/\partial t \qquad \mathbf{B} = \nabla \times \mathbf{A}$$

Which automatically satisfy $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + \partial\mathbf{B}/\partial t = 0$ due to vector identities.

But the above definitions do **not** uniquely define the magnetic vector potential because by definition we can arbitrarily add curl-free components to the magnetic potential **A** without changing the magnetic field **B**. Thus, there is a <u>degree of freedom</u> available when choosing **A**, which is known as **gauge invariance**.

Physicists often use the **Lorenz gauge condition** (Lorenz gauge, Lorenz condition) : $(\partial\varphi/\partial t)/c^2 + \nabla \cdot \mathbf{A} = 0$ which provides a relationship between A and φ to simplify the remaining Maxwell's equations :

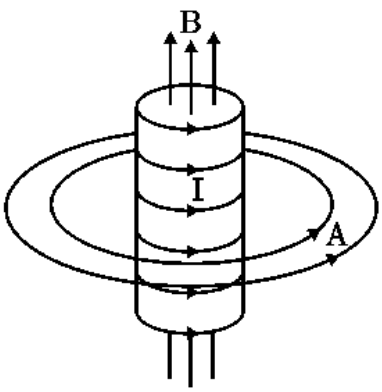
$$(\partial^2\varphi/\partial t^2)/c^2 - \nabla^2\varphi = \rho/\varepsilon_0 \qquad (\partial^2\mathbf{A}/\partial t^2)/c^2 - \nabla^2\mathbf{A} = \mu_0\mathbf{J}$$

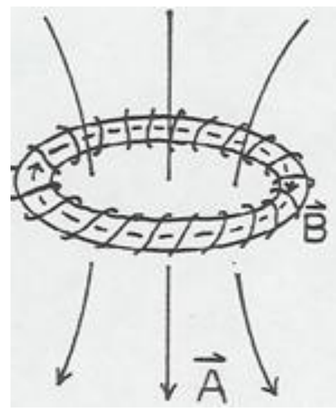
Or

$$\Box^2 \varphi = \rho/\varepsilon_0 \qquad \Box^2 \mathbf{A} = \mu_0 \mathbf{J}$$

And using **A** and φ, the Lorentz force becomes : $\mathbf{F} = q\,[-\nabla\varphi - (\partial\mathbf{A}/\partial t) + \mathbf{v} \times (\nabla\times\mathbf{A})]$

The relationship between current, **B**, and **A** are shown in the following figures :





<u>Special Relativity</u>

We can actually derive Maxwell's equations by starting from Coulomb's Law for source charges at rest and applying the Lorentz transformation for charges in motion. So a stationary charge in one frame creates an <u>electric field</u> in that frame, but to a frame travelling by the charge (so the charge appears to be moving relative to that frame), it creates a <u>magnetic field</u>! Once we have Maxwell's equations in the correct format for special relativity, we can calculate how EM fields transform between Lorentz frames and calculate fields when charges are moving at relativistic speeds.

In special relativity, everything needs to be a four-vector. The formulas presented here may differ between authors due to their choice of metric signature and whether they prefer a contravariant or covariant format.

The four-current is defined by $J^\alpha = (c\rho, \mathbf{J}) = \gamma\rho(c, \mathbf{u})$ where ρ and **J** are as before. Note that **J** represents an ordinary 3-vector, while J^α represents a 4-vector.

Then the continuity equation can be written as : $\partial_\alpha J^\alpha = 0$ and because J^α is a four-vector, this equation is invariant, and so electric charge is an invariant Lorentz scalar!

The electromagnetic four-potential, electromagnetic potential, four vector potential, or four-potential is :

$$A^\alpha = (\varphi/c, \mathbf{A})$$

Then the Lorenz gauge condition becomes : $\partial_\alpha A^\alpha = 0$ which is invariant under Loren**t**z transformations.

And Maxwell's equations may be written as one equation : $\Box^2 A^\alpha = (\partial^2 A^\alpha/\partial t^2)/c^2 - \nabla^2 A^\alpha = \mu_0 J^\alpha$

While the scalar and vector potentials combine to create the four-potential, how do the electric and magnetic fields themselves transform under Lorentz transformations? Since they are represented by 6 components, they can't be a four-vector. But in 4-D, an antisymmetric tensor has 6 independent components. As it turns out, the E and B components combine into a Lorentz-invariant rank 2 tensor called the "**electromagnetic field tensor**", "**electromagnetic field strength**", or just "**electromagnetic tensor**" :

$$F^{\mu\nu} = \begin{bmatrix} 0 & -E_x/c & -E_y/c & -E_z/c \\ E_x/c & 0 & -B_z & B_y \\ E_y/c & B_z & 0 & -B_x \\ E_z/c & -B_y & B_x & 0 \end{bmatrix}$$

Note that $F^{\alpha\beta} = -F^{\beta\alpha}$ (→ antisymmetric)

Then for an observer moving with 4-velocity u^a, the electric field they observe is : $E_a = F_{ab}u^b$
and the magnetic field they observe is : $B_a = -\varepsilon_{ab}{}^{cd} F_{cd} u^b / 2c = -\varepsilon_{abcd} F^{cd} u^b / 2c$

The EM tensor can be defined in terms of the four-potential as :

$$F^{\alpha\beta} = \partial^\alpha A^\beta - \partial^\beta A^\alpha$$

For example : $F^{12} = \partial^1 A^2 - \partial^2 A^1 = \partial^x A^y - \partial^y A^x = -(\nabla \times \mathbf{A})_z = -B_z$

Using this definition, Maxwell's equations take the following form :

$$\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$$

Which combines $\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$ and $\nabla \times \mathbf{B} - (\partial\mathbf{E}/\partial t)/c^2 = \mu_0 \mathbf{J}$

And

$$\partial_\mu F_{\alpha\beta} + \partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha} = 0$$

Which combines $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} + \partial\mathbf{B}/\partial t = 0$. Although there appear to be 4x4x4=64 equations in this tensor formula, it actually reduces to just four independent equations. Using the antisymmetry of the electromagnetic field and vector identities, all the equations except for those with μ,α,β = 0,1,2 or 1,2,3 or 2,3,0 or 3,0,1 can be shown to either reduce to an identity or be redundant.

Example – take $\partial_\alpha F^{\alpha\beta} = \mu_0 J^\beta$ for β=0 :

$$\partial_\alpha F^{\alpha 0} = \mu_0 J^0$$
$$\partial_0 F^{00} + \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = \mu_0 J^0$$
$$\partial_t(0) + \partial_x E_x/c + \partial_y E_y/c + \partial_z E_z/c = \mu_0 c\rho$$
$$\partial_x E_x + \partial_y E_y + \partial_z E_z = \mu_0 c^2 \rho$$
$$\nabla \cdot \mathbf{E} = \rho/\varepsilon_0$$

Example – take $\partial_\mu F_{\alpha\beta} + \partial_\alpha F_{\beta\mu} + \partial_\beta F_{\mu\alpha} = 0$ for μ=1, α=2, β=3 :

$$\partial_1 F_{23} + \partial_2 F_{31} + \partial_3 F_{12} = 0$$
$$-\partial_x B_x - \partial_y B_y - \partial_z B_z = 0$$

(note : $F_{\mu\nu} = g_{\mu i} g_{\nu j} F^{ij}$ as usual)

$$\nabla \cdot \mathbf{B} = 0$$

In special relativity, the Lorentz 4-force is :

$$f^{\mu} = dp^{\mu}/d\tau = q\,F^{\mu\nu}\,u_{\nu}$$

Example – take $f^{\mu} = q\,F^{\mu\nu}\,u_{\nu}$ for $\mu=1$:

$$f^{1} = q\,F^{1\nu}\,u_{\nu}$$
$$f^{1} = q(F^{10}\,u_0 + F^{11}\,u_1 + F^{12}\,u_2 + F^{13}\,u_3)$$
$$f^{1} = q\gamma(F^{10}c + F^{11}\,v_1 + F^{12}\,v_2 + F^{13}\,v_3)$$
$$f^{x} = q\gamma(cE_x/c + 0\,v_x - B_z\,v_y + B_y\,v_z)$$
$$f^{x} = q\gamma(E_x - B_z v_y + B_y v_z)$$
$$f^{x} = q\gamma[E_x + (\mathbf{v} \times \mathbf{B})_x]$$

And so

$$(f^{x}, f^{y}, f^{z}) = \gamma q[\mathbf{E} + \mathbf{v} \times \mathbf{B}]$$

Example – take $f^{\mu} = q\,F^{\mu\nu}\,u_{\nu}$ for $\mu=0$:

$$f^{0} = q(F^{00}\,u_0 + F^{01}\,u_1 + F^{02}\,u_2 + F^{03}\,u_3)$$
$$f^{0} = q\gamma(F^{00}c + F^{01}\,v_1 + F^{02}\,v_2 + F^{03}\,v_3)$$
$$P(\text{power}) = q\gamma(0\,c - v_1E_x/c - v_2E_y/c - v_3E_x/c)$$
$$P = -\gamma q\,\mathbf{E}\cdot\mathbf{v}$$

General Relativity

All that is needed to go from flat spacetime to curved spacetime is to replace $\eta_{\mu\nu}$ with $g_{\mu\nu}$ and ∂ with ∇.

Then the Lorenz gauge condition becomes : $\nabla_{\alpha}A^{\alpha} = 0$

And in terms of the four-potential, Maxwell's equations become : $\nabla_{\alpha}\nabla^{\alpha}A^{\beta} - R^{\beta}{}_{\lambda}A^{\lambda} = \mu_0\,J^{\beta}$

The electromagnetic tensor becomes : $F_{\alpha\beta} = \nabla_{\alpha}A_{\beta} - \nabla_{\beta}A_{\alpha}$

And only when there are no sources *in* the region of interest, $\nabla_{\beta}F^{\alpha\beta} = 0$

In terms of the EM tensor, Maxwell's equations become :

$$\nabla_{\alpha}\,F^{\alpha\beta} = \mu_0 J^{\beta}$$
$$\nabla_{\mu}F_{\alpha\beta} + \nabla_{\alpha}F_{\beta\mu} + \nabla_{\beta}F_{\mu\alpha} = 0$$

Where the continuity equation is incorporated into the first equation :

$$\nabla_{\beta}\nabla_{\alpha}\,F^{\alpha\beta} = \mu_0\nabla_{\beta}\,J^{\beta}$$

But $\nabla_{\beta}\nabla_{\alpha}\,F^{\alpha\beta} = 0$ (by identities) so $\nabla_{\beta}\,J^{\beta} = 0$

And the Lorentz force is still : $f_{\alpha} = q\,F_{\alpha\nu}\,u^{\nu}$

If there is an electromagnetic field present in the spacetime under consideration, then the total stress-energy-momentum tensor includes an EM term :

$$T^{\mu\nu} = (F^{\mu\alpha}F_{\alpha}{}^{\nu} - g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}/4) \, / \, \mu_0$$

Or as the mixed type

$$T^{\mu}{}_{\nu} = (F^{\mu\alpha}F_{\alpha\nu} - \delta^{\mu}{}_{\nu}F_{\alpha\beta}F^{\alpha\beta}/4) \, / \, \mu_0$$

$T^{\mu\nu}$ can also be represented as :

$$T^{\mu\nu} = \begin{bmatrix} \frac{1}{2}(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2) & S_x/c & S_y/c & S_z/c \\ S_x/c & -\sigma_{xx} & -\sigma_{xy} & -\sigma_{xz} \\ S_y/c & -\sigma_{yx} & -\sigma_{yy} & -\sigma_{yz} \\ S_z/c & -\sigma_{zx} & -\sigma_{zy} & -\sigma_{zz} \end{bmatrix},$$

where **S** is the Poynting vector :

$$\vec{S} = \frac{1}{\mu_0}\vec{E} \times \vec{B}$$

and σ_{ij} is the Maxwell stress tensor :

$$\sigma_{ij} = \epsilon_0 E_i E_j + \frac{1}{\mu_0}B_i B_j - \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)\delta_{ij}$$

If $J^a \neq 0$, then E^{ab} alone is not conserved, but the total stress-energy-momentum of the EM field *and* the (possibly moving) charged matter creating the EM field is still conserved :

$$\nabla_{\beta} T^{\alpha\beta} + f^{\alpha} = 0$$

This equation is equivalent to the following conservation laws in 3-D vector form :

$$\frac{\partial u_{em}}{\partial t} + \vec{\nabla} \cdot \vec{S} + \vec{J} \cdot \vec{E} = 0$$ is Poynting's Theorem (for μ=0 in above equation)

$$\frac{\partial \vec{p}_{em}}{\partial t} - \vec{\nabla} \cdot \sigma + \rho\vec{E} + \vec{J} \times \vec{B} = 0$$

where

$$u_{em} = \frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2$$ is the electromagnetic energy density

$$\vec{p}_{em} = \frac{\vec{S}}{c^2}$$ is the electromagnetic momentum density